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Geometry of Hamiltonian n-vector fields in multisymplectic field theory

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Abstract

Multisymplectic geometry—which originates from the well known De Donder–Weyl (DW) theory—is a natural framework for the study of classical field theories. Recently, two algebraic structures have been put forward to encode a given theory algebraically. Those structures are formulated on finite dimensional spaces, which seems to be surprising at first.

In this paper, we investigate the correspondence of Hamiltonian functions and certain antisymmetric tensor products of vector fields. The latter turn out to be the proper generalisation of the Hamiltonian vector fields of classical mechanics. Thus we clarify the algebraic description of solutions of the field equations.

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1. Introduction

It has long been known that the appropriate language for classical field theories is the formalism of jet bundles. Within this framework, the Langrangean variational principle can be formulated and the Euler–Lagrange equations can be derived. Furthermore, the theorem of Noether [13] which relates symmetries of the Lagrange density and conserved quantities can be given a geometrical interpretation.

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In this paper, we consider first order field theories, i.e. theories which are defined by a Lagrange density that depends on the fields and their first derivatives only. In this case the field equations are second order partial differential equations. These equations can be transformed into a Hamiltonian system on an infinite dimensional space—this is the canonical Hamiltonian formalism on the space of initial data. One has to distinguish a time direction in order to define a conjugate momentum for every field coordinate. This results in breaking Lorentz covariance.

Alternatively, there is a framework that can be formulated on finite dimensional geometries (for a detailed review, we refer to [5]). Moreover, space and time directions are treated in a covariant way. This approach is known under the name De Donder–Weyl (DW) formalism or covariant Hamiltonian theory. The paper at hand will stay within this framework.

In contrast to classical mechanics, it introduces more than one conjugated momentum variable for each degree of freedom. Using a covariant generalisation of the Legendre transformation of classical mechanics, one can perform the transition from the second order Euler–Lagrange equations to the first order DW equations. The latter are formulated for sections of what is called multisymplectic phase space, i.e. smooth maps from the base manifold into that space. Keeping in mind that trajectories in classical mechanics are maps from the time axis to phase space, the treatment in the DW formalism is a generalisation to more than one evolution parameter.

Only recently two algebraic structures have been proposed that encode the up to now geometrical picture of (partial) differential equations for sections. While Forger and Römer [4] work on the extended multisymplectic phase space $\mathcal{P}$ that generalises the doubly extended phase space of time dependent symplectic mechanics, Kanatchikov [7,8] uses a space that has one dimension less than $\mathcal{P}$ and can be interpreted as the parameter space of hypersurfaces of constant DW Hamiltonians. This space will be denoted $\tilde{\mathcal{P}}$ for the rest of this paper.

Note that both $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are multisymplectic manifolds in the sense of Martin ([11], in which there is a generalised Darboux theorem) only for very special bundles $\mathcal{E}$. Rather, we will use the term in the more general sense of a manifold with a closed, non-degenerate form [1,5].

The main advantage as compared to ordinary field theoretical Poisson structures is that the underlying spaces $\mathcal{P}$ and $\tilde{\mathcal{P}}$ both are finite dimensional. The price one has to pay for this is that there is more than one conjugated momentum associated with each coordinate degree of freedom. Up to now, this has been an obstacle to the application of the standard quantisation programme.

It remains to understand in which sense the algebraic structures describe the solutions of the field equations, i.e. the states of the system under consideration.

The idea is that in the case of mechanics there is a correspondence between vector fields and curves in phase space. The former can be viewed as derivations on the algebra of smooth functions on the phase space, and can be described by functions that act via the Poisson bracket if the vector fields are Hamiltonian. In multisymplectic geometry, on the other hand, curves are replaced by sections of some bundle which consequently are higher dimensional. Therefore, they are described by a set of tangent vectors at every point which span a distribution on the extended multisymplectic phase space, i.e. that specify some subspace of the tangent space at every point. Furthermore, if the distribution is of constant

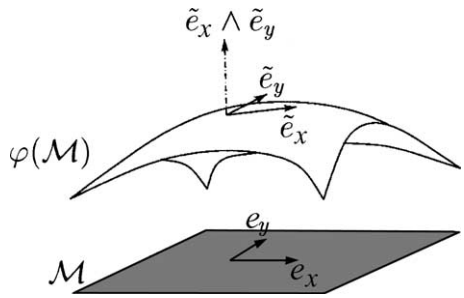


Fig. 1. The lifted vectors $\tilde{e}_x$ and $\tilde{e}_y$ span the tangent space at a point of the section Φ. The two vectors can be combined to give a 2-vector $\tilde{e}_x \wedge \tilde{e}_y$ that describes this tangent space.

rank (i.e. if the sections do not have kinks), one can pick (in a smooth way) a basis of the specified subspace in the tangent bundle at every point and combine the basis vectors using the antisymmetric tensor product of vectors to obtain a multivector field, see Fig. 1. This multivector field is unique up to multiplication by a function and of constant tensor grade.

Kanatchikov was the first to note that the fundamental relationship of symplectic geometry between Hamiltonian vector fields X_f and functions f given by

$$X_f \lrcorner \omega = \mathrm{d}f, \tag{1}$$

where ω denotes the symplectic 2-form, can be generalised to cover the multisymplectic case, in which ω is the multisymplectic form, a closed non-degenerate $(n+1)$-form (n being the dimension of space–time), f is an r-form and X_f has to be a multivector field of tensor grade $(n-r)$. Consequently, if f is a function then X_f has to be an n-vector field, i.e. a multivector field of tensor grade n. This is a good candidate to describe distributions that yield sections of the fibre bundle. The link between Hamiltonian n-vector fields and solutions of the field equations has already been indicated by Kanatchikov in [7]. Moreover, the sense in which multivector fields are related to distributions seems to be folklore and is written out explicitly in the work by Echeverría-Enríquez et al. [3], see Appendix A of this paper. However, both use the smaller multisymplectic phase space $\tilde{\mathcal{P}}$ which requires the choice of a connection [14]. Moreover, we will show in Theorem 3 that for typical cases in field theory the generalisation of (1) to $\tilde{\mathcal{P}}$ does not admit the interpretation of X_f to define a distribution. Instead, one has to go over to the extended multisymplectic phase space $\mathcal{P}$. This is not in contradiction to the results established by Echeverría-Enríquez et al. since they consider an equation different from (1), namely

$$X_f \lrcorner (\omega - \mathrm{d}f \wedge \mathrm{d}^n x) = 0, \tag{2}$$

where $\mathrm{d}^n x$ is a volume form on space–time (for non-trivial fibre bundles, terms containing a connection appear in addition). Therefore, although their investigation proceeds along similar lines as this paper, the results cannot be taken over to the case of multisymplectic geometry.

The structure of this paper is as follows. Section 2 reviews the basic notions needed for this paper. In particular, the multisymplectic phase spaces $\mathcal{P}$ and $\tilde{\mathcal{P}}$ and the multisymplectic forms on them are defined and Hamiltonian forms and Hamiltonian multivector fields are

introduced. Section 3 contains the main part of this paper. We will establish the link between solutions of the field equations and multivector fields associated to some appropriately chosen function in three steps. Firstly, we show that a certain class of functions on $\mathcal{P}$ admits Hamiltonian n-vector fields that define distributions. Secondly, we show that the leaves of those distributions, should they exist, are solutions to the field equations that correspond to the Hamiltonian function which has been chosen in the first place. Thirdly, we investigate under which conditions the distributions defined by the Hamiltonian n-vector fields are integrable. It will turn out that additional input is needed to answer the latter question as there is a considerable freedom to choose a Hamiltonian n-vector field for a given Hamiltonian function. This additional input is provided by a covariant version of the Hamilton–Jacobi equation. In the end, we will show that the construction cannot be taken over to $\tilde{\mathcal{P}}$.

2. Multisymplectic geometry

2.1. DW equations and multisymplectic phase spaces

Usually, classical field theories are formulated as variational problems for the fields φ—which are sections of some fibre (vector) bundle $\mathcal{E}$ over an n-dimensional base manifold (space–time) $\mathcal{M}$—and some Lagrange density $\mathcal{L}$. We will assume $\mathcal{M}$ to be orientable. $\mathcal{L}$ is a function of the fields and its first derivatives, and one is looking for extremal points of the action functional

$$\mathcal{S}(\varphi) = \int_{\mathcal{M}} \mathrm{d}^n x \mathcal{L}(x, \varphi(x), \varphi'(x)). \tag{3}$$

Mathematically, $\mathcal{L}$ is a function on the first jet bundle $\mathfrak{J}^1\mathcal{E}$ to $\mathcal{E}$ [5,10,16]. It is well known that the extremal points of this functional can be found by solving the field equations—the celebrated Euler–Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial_\mu \varphi^A}(\varphi(x)) - \frac{\partial \mathcal{L}}{\partial \varphi^A}(\varphi(x)) \right) = 0. \tag{4}$$

Here, as in all what follows, $\mu, \nu, \rho, \ldots = 1, \ldots, n$ label coordinates on $\mathcal{M}$, while $A, B, C, \ldots = 1, \ldots, N$ stand for those on the fibres of $\mathcal{E}$.

If the Lagrange density fulfils some regularity condition, the Euler–Lagrange equations can be seen to be equivalent to a set of first order equations (cf. [15])

$$\frac{\partial \mathcal{H}}{\partial \pi^\mu_A}(x, \varphi(x), \pi(x)) = \partial_\mu \varphi^A(x), \qquad \frac{\partial \mathcal{H}}{\partial \varphi^A}(x, \varphi(x), \pi(x)) = -\partial_\mu \pi^\mu_A(x) \tag{5}$$

for the DW Hamiltonian $\mathcal{H}$,

$$\mathcal{H} = \pi^\mu_A \partial_\mu \varphi^A - \mathcal{L}. \tag{6}$$

In these equations, the polymomenta π^μ_A are defined as derivatives of the Lagrange density by the field derivatives,

$$\pi^\mu_A = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^A}. \tag{7}$$

The regularity conditions to $\mathcal{L}$ ensure that these equations can be used to express the field derivatives $\partial_\mu \varphi^A$ in terms of the fields φ and the polymomenta.

So far we have used sections $\varphi(x)$, etc. to formulate the equations of motion but it is useful to consider functions like the DW Hamiltonian $\mathcal{H}$ without evaluating them on fields $\varphi(x)$, etc. To this end, let us introduce coordinates v^A for fields, v^A_μ for their space–time derivatives and p^μ_A for the polymomenta functions (7). To condense notation, we will write derivatives w.r.t. the fields φ^A as ∂_A, while those w.r.t. the polymomenta π^μ_A will be denoted by ∂^A_μ. Together with an additional coordinate p, the set of variables

$$(x^\mu, v^A, p^\mu_A, p) \tag{8}$$

labels locally the extended multisymplectic phase space $\mathcal{P}$. Derivatives by this extra coordinate p, which itself can be interpreted as the DW energy variable, will be denoted by ∂. Geometrically, $\mathcal{P}$ is the (affine) dual of the first jet bundle $\mathfrak{J}^1\mathcal{E}$, i.e. the space of fields and velocities. One can show that the choice of a local chart of $\mathcal{E}$ induces coordinates on $\mathcal{P}$. The set of coordinates

$$(x^\mu, v^A, p^\mu_A) \tag{9}$$

can be used to describe locally the multisymplectic phase space $\tilde{\mathcal{P}}$. There is a canonical projection from $\mathcal{P}$ to $\tilde{\mathcal{P}}$ which projects out the additional variable p. With the help of a volume form ω on the base manifold $\mathcal{M}$ we find

$$\tilde{\mathcal{P}} \overset{\omega}{\cong} (\mathfrak{V}\mathcal{E})^* \otimes T\mathcal{M}, \qquad \mathcal{P} \overset{\omega,\Gamma}{\cong} \tilde{\mathcal{P}} \oplus \mathcal{R}, \tag{10}$$

where $\mathfrak{V}\mathcal{E}$ is the vertical tangent subbundle of $\mathcal{E}$ and $\mathcal{R}$ denotes a trivial line bundle on $\mathcal{E}$. For the latter isomorphism, a connection Γ of $\mathcal{E}$ is needed in addition [14]. Note that the tensor products are understood pointwise on $\mathcal{E}$.

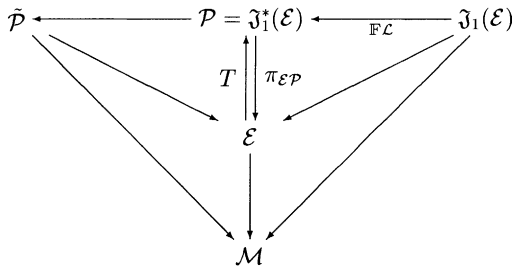
At this point it is useful to examine the special case if $\mathcal{M}$ happens to be the real axis $\mathbb{R}$, i.e. if there is only time and no space-like direction. Then, $\mathcal{E}$ is trivial, say $\mathcal{E} = \mathbb{R} \times \mathcal{Q}$, and $\mathfrak{J}^1\mathcal{E} = \mathbb{R} \times T\mathcal{Q}$. The extended multisymplectic phase space $\mathcal{P}$ then becomes $\mathcal{P} = \mathbb{R}^2 \times T^*\mathcal{Q}$, which is the doubly extended phase space of a time-dependent classical mechanical system with configuration space $\mathcal{Q}$. $\tilde{\mathcal{P}}$ is in this case the singly extended phase space. We will, however, suppress the word single in order to keep the names short.

With these spaces introduced, Eqs. (6) and (7) can be understood as a map

$$\mathbb{F}\mathcal{L} : \mathfrak{J}^1\mathcal{E} \to \mathcal{P}, \qquad (x^\mu, v^A, v^A_\mu) \to \left(x^\mu, v^A, \frac{\partial \mathcal{L}}{\partial v^A_\mu}, -\left(v^A_\mu \frac{\partial \mathcal{L}}{\partial v^A_\mu} - \mathcal{L}\right)\right), \tag{11}$$

which is known as Legendre transformation (the symbol $\mathbb{F}\mathcal{L}$ is chosen to express that it is a fibre derivation using the Lagrange density). If the Lagrange density is regular, this map defines a bijective map from $\mathfrak{J}^1\mathcal{E}$ to $\tilde{\mathcal{P}}$.

For convenience, the different spaces introduced so far will be displayed in a diagram (the map T will be needed below).



(12)

2.2. Multisymplectic forms

It has long been known that there are generalisations of symplectic geometry to field theory. The crucial observation which lead to the development of those generalisations was that in field theory, solutions are sections (of some fibre bundle), while in classical mechanics, solutions are curves. Hence, one can try to treat the sections as higher dimensional analogues of curves, i.e. treat the space-like coordinates of the fields under investigation as additional evolution parameters, cf. (5). These efforts culminated in the discovery of the multisymplectic form, an $(n+1)$-form which is to replace the symplectic 2-form. The multisymplectic $(n+1)$-form is defined on the doubly extended multisymplectic phase space $\mathcal{P}$. In coordinates, it is given by

$$\Omega_{(x,v,\vec{p},p)} = \mathrm{d}v^A \wedge \mathrm{d}p^\mu_A \wedge \mathrm{d}_\mu x - \mathrm{d}p \wedge \mathrm{d}x. \tag{13}$$

Here, $\vec{p}$ is a shorthand notation for the polymomenta p^μ_A. We refer to the work of Gotay et al. [5] for a detailed review. Note that Ω is an exact form. Using Ω, one defines pairs of Hamiltonian multivector fields X, $X \in \Gamma(\Lambda^* T\mathcal{P})$, and Hamiltonian forms H by the equation

$$X \lrcorner \Omega = \mathrm{d}H. \tag{14}$$

From degree counting, it is immediate that H can be a form of maximal degree $(n-1)$. If H is a homogeneous form of degree $|H|$, then the corresponding Hamiltonian multivector field X has to be an $(n-|H|)$-vector field. Observe that—in contrast to classical mechanics—neither side is uniquely defined, although Ω is non-degenerate on vector fields.

Because of the peculiar combination of field and polymomentum forms in (13) the dependence of a Hamiltonian form on the coordinates p^μ_A is subject to strong restrictions. Unless H is a function, it has to be a polynomial of maximal degree $|H|$ in the polymomenta [7,14]. There are additional restrictions to the specific form of that polynomial dependence.

On the multisymplectic phase space $\tilde{\mathcal{P}}$, there is no such canonical $(n+1)$-form, but one can separate the first summand of (13) and transport it to $\tilde{\mathcal{P}}$. The resulting $(n+1)$-form is called vertical multisymplectic $(n+1)$-form. Its coordinate expression is

$$\Omega_{\Gamma(x,v,\vec{p},p)} = \mathrm{d}v^A \wedge \mathrm{d}p^\mu_A \wedge \mathrm{d}_\mu x + f_A(x,v)\,\mathrm{d}v^A \wedge \mathrm{d}^n x + g^A_\mu(x,v,\vec{p})\,\mathrm{d}p^\mu_A \wedge \mathrm{d}^n x. \tag{15}$$

For the construction of Ω_Γ, a connection of the fibre bundle $\mathcal{E}$ has to be chosen. This choice creates the last two terms in the above formula. Their precise expressions will not be important for what follows (for them, we refer to [3]). Using Γ again, one can define a vertical exterior derivative d_Γ on $\tilde{\mathcal{P}}$, i.e. a map with square zero that takes derivatives with respect to the vertical directions only, i.e. w.r.t. the v^A and p^μ_A variables.[1] Combining Ω_Γ and d_Γ, one can ask for solutions (X_H, H) of

$$X_H \lrcorner \Omega_\Gamma = \mathrm{d}_\Gamma H. \tag{16}$$

In this case, H is called Hamiltonian form on $\tilde{\mathcal{P}}$. Again, the polymomentum dependence of H is subject to restrictions unless H is a function.

3. Hamiltonian n-vector fields

3.1. Decomposition of Hamiltonian n-vector fields

It is a well-known fact [10] that submanifolds can be described by (integrable) distributions, i.e. the determination of some subspace of the tangent bundle at every point of a manifold. In the appendix, we show that such subspaces of dimension n are in exact correspondence to the decomposable[2] n-vector fields, i.e. such vector fields that can be written (locally) as the anti-symmetrised tensor product of n distinct vector fields, cf. Fig. 1. As explained in the appendix, n-dimensional subspaces of $T\mathcal{P}$ are described by such n-vector fields that can be written as the n-fold antisymmetric tensor product of vector fields. Therefore, we will examine for which Hamiltonian n-vector fields this property can be achieved.

Theorem 1. *Let $H \in \mathcal{C}^\infty(\mathcal{P})$ be a function on the multisymplectic phase space. If H is of the particular form*

$$H(x, v, \vec{p}, p) = -\mathcal{H}(x, v, \vec{p}) - p, \tag{17}$$

where $\mathcal{H}$ is an arbitrary function not depending on p, then there is a decomposable Hamiltonian vector field X corresponding to H.

Remark. The condition on H can be formulated without referring to coordinates. As $\mathcal{P}$ is an affine bundle over $\tilde{\mathcal{P}}$ with a trivial associated line bundle it carries a fundamental vector field ξ, the derivation w.r.t. the p-direction. The condition on H is then $\xi(H) = -1$. It will become clear in the next section why we distinguish the particular p-dependence. Note in particular that this property does not depend on the coordinate system used.

[1] Acting on the coordinate functions v^A and p^μ_A, d_Γ yields the corresponding 1-forms that are vertical w.r.t. the connection $\bar{\Gamma}$ that can be induced from Γ and a connection on $\mathcal{M}$, cf. [14] for details.

[2] There seems to be no standard terminology for the special elements in the n-fold antisymmetric tensor product of a vector space V that are of the form

$$Z_1 \wedge \cdots \wedge Z_n \in \Lambda^n(V), \qquad Z_\mu \in V.$$

In [6, Chapter 3], they are called *decomposable*, while in [2, Section V.1.06], the word *simple* is used for them.

Proof. When expressed in coordinates, the condition for X to be a Hamiltonian n-vector field to some Hamiltonian $H \in \mathcal{C}^\infty(\mathcal{P})$,

$$X \lrcorner \Omega = \mathrm{d}H, \tag{18}$$

where

$$\begin{aligned} X = {} & \frac{1}{n!} X^{\nu_1 \cdots \nu_n} \partial_{\nu_1} \cdots \partial_{\nu_n} + \frac{1}{(n-1)!} X^{A \nu_1 \cdots \nu_{n-1}} \partial_A \partial_{\nu_1} \cdots \partial_{\nu_{n-1}} \\ & + \frac{1}{(n-1)!} X_A^{\sigma \nu_1 \cdots \nu_{n-1}} \partial_\sigma^A \partial_{\nu_1} \cdots \partial_{\nu_{n-1}} + \frac{1}{(n-1)!} X_0^{\nu_1 \cdots \nu_{n-1}} \partial \partial_{\nu_1} \cdots \partial_{\nu_{n-1}} \\ & + \frac{1}{(n-2)!} X_A^{\sigma B \nu_1 \cdots \nu_{n-2}} \partial_\sigma^A \partial_B \partial_{\nu_1} \cdots \partial_{\nu_{n-2}} + \text{terms of higher vertical order}, \end{aligned}$$

amounts to

$$\begin{aligned} & \partial_A H = \frac{(-)^n}{(n-1)!} X_A^{\mu \nu_1 \cdots \nu_{n-1}} \epsilon_{\mu \nu_1 \cdots \nu_{n-1}}, \qquad \partial_\mu^A H = -\frac{(-)^n}{(n-1)!} X^{A \nu_1 \cdots \nu_{n-1}} \epsilon_{\mu \nu_1 \cdots \nu_{n-1}}, \\ & \partial_\mu H = -\frac{1}{(n-2)!} X_A^{\sigma A \nu_1 \cdots \nu_{n-2}} \epsilon_{\sigma \nu_1 \cdots \nu_{n-2} \mu} - \frac{1}{(n-1)!} X_0^{\nu_1 \cdots \nu_{n-1}} \epsilon_{\nu_1 \cdots \nu_{n-1} \mu}, \\ & \partial H = \frac{(-)^{n+1}}{n!} X^{\nu_1 \cdots \nu_n} \epsilon_{\nu_1 \cdots \nu_n}. \end{aligned} \tag{19}$$

Now let Z_μ be a set of n-vectors,

$$Z_\mu = (Z_\mu)^\nu \partial_\nu + (Z_\mu)^A \partial_A + (Z_\mu)_A^\nu \partial_\nu^A + (Z_\mu)_0 \partial. \tag{20}$$

The wedge product of all Z_μ, $\mu = 1, \ldots, n$ gives (in obvious cases we will omit the symbol $\wedge$)

$$\begin{aligned} Y = Z_1 \wedge \cdots \wedge Z_n = {} & (Y_1)^{\mu_1} \cdots (Y_n)^{\mu_n} \epsilon_{\mu_1 \cdots \mu_n} \partial_{x^1} \cdots \partial_{x^n} \\ & + \sum_{\mu=1}^n (-)^{\mu+1} (Z_\mu)^A (Z_1)^{\nu_1} \cdots \widehat{(Z_\mu)^{\nu_\mu}} \cdots (Z_n)^{\nu_n} \partial_A \partial_{\nu_1} \cdots \widehat{\partial_{\nu_\mu}} \cdots \partial_{\nu_n} \\ & + \sum_{\mu=1}^n (-)^{\mu+1} (Z_\mu)_A^\sigma (Z_1)^{\nu_1} \cdots \widehat{(Z_\mu)^{\nu_\mu}} \cdots (Z_n)^{\nu_n} \partial_\sigma^A \partial_{\nu_1} \cdots \widehat{\partial_{\nu_\mu}} \cdots \partial_{\nu_n} \\ & + \sum_{\mu<\nu} (-)^{\mu+\nu} ((Z_\mu)^A (Z_\nu)_B^\sigma - (Z_\nu)^A (Z_\mu)_B^\sigma) (Z_1)^{\rho_1} \cdots \widehat{(Z_\mu)^{\rho_\mu}} \cdots \widehat{(Z_\nu)^{\rho_\nu}} \cdots \\ & \times (Z_n)^{\rho_n} \partial_\sigma^B \partial_A \partial_{\rho_1} \cdots \widehat{\partial_{\rho_\mu}} \cdots \widehat{\partial_{\rho_\nu}} \cdots \partial_{\rho_n} + \text{terms of higher vertical order}. \end{aligned} \tag{21}$$

(In this calculation, a hat on top of a symbol means the omission of that symbol.)

Comparing this to X, one finds in the first place

$$(Z_1)^{\mu_1} \cdots (Z_n)^{\mu_n} \epsilon_{\mu_1 \cdots \mu_n} = \frac{1}{n!} X^{\mu_1 \cdots \mu_n} \epsilon_{\mu_1 \cdots \mu_n} = (-)^{n+1} \partial H = (-1)^n \tag{22}$$

The n-vectors Z_μ of (20) define a linear map from $T\mathcal{M}$ to $T\mathcal{P}$ for every point on $\mathcal{P}$. Let us denote this map by Z. Using the canonical projection $\pi_0^\star$ of $\mathcal{P}$ onto $\mathcal{M}$ we obtain a

map $T\pi_0^{\star}\circ Z$ from $T\mathcal{M}$ to itself. Eq. (22) describes the determinant of this map in the coordinates chosen. There is a straightforward solution, namely

$$(Z_\mu)^\nu = -\delta^\nu_\mu. \tag{23}$$

It is clear that if $\partial H = 0$ at some point the components $(Z_\mu)^\nu$ of the vector fields Z_μ cannot be linearly independent Z_μ one from another and hence cannot span the n-dimensional tangent space on $\mathcal{M}$.

Comparing the next terms of Y and X one obtains

$$\begin{aligned}
&\frac{1}{(n-1)!}X^{A\nu_1\cdots\nu_{n-1}}\epsilon_{\nu_1\cdots\nu_{n-1}\rho}\\
&\quad=\sum_{\mu=1}^{n}(-)^{\mu+1}(Z_\mu)^A(Z_1)^{\nu_1}\cdots\widehat{(Z_\mu)^{\nu_\mu}}\cdots(Z_n)^{\nu_n}\epsilon_{\nu_1\cdots\widehat{\nu_\mu}\cdots\nu_n\rho},\\
&\frac{1}{(n-1)!}X_A^{\sigma\nu_1\cdots\nu_{n-1}}\epsilon_{\nu_1\cdots\nu_{n-1}\rho}\\
&\quad=\sum_{\mu=1}^{n}(-)^{\mu+1}(Z_\mu)^\sigma_A(Z_1)^{\nu_1}\cdots\widehat{(Z_\mu)^{\nu_\mu}}\cdots(Z_n)^{\nu_n}\epsilon_{\nu_1\cdots\widehat{\nu_\mu}\cdots\nu_n\rho}.
\end{aligned}\tag{24}$$

Now let $(Z_\mu)^\nu$ be given by (23). Then

$$\begin{aligned}
\partial^A_\rho H &= \frac{1}{(n-1)!}X^{A\nu_1\cdots\nu_{n-1}}\epsilon_{\nu_1\cdots\nu_{n-1}\rho} = (Z_\rho)^A - \partial_A H\\
&= \frac{1}{(n-1)!}X_A^{\rho\nu_1\cdots\nu_{n-1}}\epsilon_{\nu_1\cdots\nu_{n-1}\rho} = (Z_\rho)^\rho_A,
\end{aligned}\tag{25}$$

which obviously is satisfied by

$$(Z_\mu)^A = \partial^A_\mu H, \qquad (Z_\mu)^\nu_A = -\frac{1}{n}\delta^\nu_\mu\partial_A H + (Z'_\mu)^\nu_A, \tag{26}$$

where the $(Z'_\mu)^\nu_A$ are arbitrary functions that satisfy

$$(Z'_\mu)^\mu_A = 0.$$

Note that the momentum directions of Z_μ are not given uniquely. In particular, there are no conditions on the off-diagonal terms $(Z_\mu)^\nu_A$, $\mu\neq\nu$. It remains to determine the components $(Z_\mu)_0$, but this can be done using the third line in (19). Indeed, further comparison of (18)–(21) yields

$$-\frac{1}{(n-2)!}X_A^{\sigma B\nu_1\cdots\nu_{n-2}}\epsilon_{\rho_1\nu_1\cdots\nu_{n-2}\rho_2} = (Z_{\rho_1})^B(Z_{\rho_2})^\sigma_A - (Z_{\rho_2})^B(Z_{\rho_1})^\sigma_A. \tag{27}$$

Using (19) we obtain for a special contraction

$$\begin{aligned}
\partial_\mu H &= -\frac{1}{(n-2)!}X_A^{\sigma A\nu_1\cdots\nu_{n-2}}\epsilon_{\sigma\nu_1\cdots\nu_{n-2}\mu} - \frac{1}{(n-1)!}X_0^{\nu_1\cdots\nu_{n-1}}\epsilon_{\nu_1\cdots\nu_{n-1}\mu}\\
&= -((Z_\mu)^A(Z_\nu)^\nu_A - (Z_\nu)^A(Z_\mu)^\nu_A) - (Z_\mu)_0.
\end{aligned}\tag{28}$$

This yields an expression for $(Z_\mu)_0$ in terms of the other components of Z_μ.

Eq. (22) shows that if $\partial H \neq 0$ then the Z_μ are linearly independent (as their horizontal components are). Hence $Y \neq 0$. Moreover, the components of Z_μ have been determined using all of (19). Thus Y is a Hamiltonian vector field to H. □

3.2. Solutions define decomposable Hamiltonian n-vectors

As a next step, we ask what the Hamiltonian 0-forms H have to do with the DW Hamiltonian $\mathcal{H}$. Their relation is already indicated in the notation of (17) and can be guessed further from (25).

Theorem 2. *Let $\gamma = (\varphi, \pi)$ be a solution of the DW equation (5) for some DW Hamiltonian $\mathcal{H}$. The tangent space of the image of γ defines an n-vector field which is Hamiltonian with respect to the function H given by (17).*

Remark. From Lemma A.1, we know that an n-vector X is decomposable if and only if there are n linearly independent vectors Z_μ which satisfy $Z_\mu \wedge X = 0$. This implies for the Hamiltonian n-vector fields X of the given function H

$$0 = (X \wedge Z_\mu) \lrcorner \Omega = Z_\mu \lrcorner \mathrm{d}H. \tag{29}$$

Combining (6) and (11) we note that H vanishes on sections γ that satisfy the DW equations. Therefore, it is natural to expect that $Z_1 \wedge \cdots \wedge Z_n$ is proportional to a Hamiltonian n-vector field X if the vector fields Z_μ are lifts by γ.

Proof. In local coordinates, the section γ is given by

$$\gamma(x) = (\varphi^A(x), \pi^\nu_A(x), -\mathcal{H}(x, \varphi(x), \pi(x))). \tag{30}$$

Let ∂_μ, $\mu = 1, \ldots, n$, be a basis of $T_m\mathcal{M}$. Their respective lifts Z_μ to $T\mathcal{P}$ via γ are given by

$$\begin{aligned} Z_\mu &= \partial_\mu + \partial_\mu\varphi^A \partial_A + \partial_\mu \pi^\nu_A \partial^A_\nu - [\partial_\mu \mathcal{H} + \partial_A \mathcal{H} \partial_\mu \varphi^A + \partial^A_\sigma \mathcal{H} \partial_\mu \pi^\sigma_A]\partial \\ &= \partial_\mu + \partial^A_\mu \mathcal{H} \partial_A + \partial_\mu \pi^\nu_A \partial^A_\nu - [\partial_\mu \mathcal{H} + \partial_A \mathcal{H} \partial^A_\mu \mathcal{H} + \partial^A_\sigma \mathcal{H} \partial_\mu \pi^\sigma_A]\partial. \end{aligned} \tag{31}$$

Note that the vector fields Z_μ are not defined on all of $\mathcal{P}$. Rather, they are given on the image of some local region in $\mathcal{M}$ under γ only.

Let X be a Hamiltonian n-vector field and $\tilde{Z}_1 \wedge \cdots \wedge \tilde{Z}_n$ be a decomposition of it. Using the calculations of the preceding section, we conclude from Eq. (23)

$$(\tilde{Z}_\mu)^\nu = -\delta^\nu_\mu = -(Z_\mu)^\nu, \tag{32}$$

while from (26) it follows that

$$(\tilde{Z}_\mu)^A = \partial^A_\mu H = -\partial^A_\mu \mathcal{H} = -(Z_\mu)^A, \quad (\tilde{Z}_\mu)^\mu_A = -\partial_A H = \partial_A \mathcal{H} = -(Z_\mu)^\mu_A. \tag{33}$$

Finally, we compute for the remaining component $(Z_\mu)_0$

$$(Z_\mu)_0 = -\partial_\mu \mathcal{H} - \partial_A \mathcal{H} \partial^A_\mu \mathcal{H} - \partial^A_\sigma \mathcal{H} \partial_\mu \pi^\sigma_A = \partial_\mu H + ((Z_\sigma)^\sigma_A (Z_\mu)^A - (Z_\sigma)^A (Z_\mu)^\sigma_A), \tag{34}$$

which goes over to (28) for $(\tilde{Z}_\mu)_0 = -(Z_\mu)_0$, $(\tilde{Z}_\mu)^A = -(Z_\mu)^A$, and $(\tilde{Z}_\mu)^\nu_A = -(Z_\mu)^\nu_A$. Therefore, the set of vector fields

$$\tilde{Z}_\mu = -Z_\mu, \quad \mu = 1, \ldots, n \tag{35}$$

defines a decomposition of a Hamiltonian n-vector field X of H. This proves the assertion. □

Remark. At this point a remark is in order about the peculiar form (17). It is known that the DW Hamiltonian (6) constitutes a relation among coordinates of $\mathcal{P}$ that describes the image of $\mathbb{F}\mathcal{L}$. If one wants to extract a function $\mathcal{H}_\Gamma$, the global Hamiltonian function of [3], out of it one needs to employ a connection in $\mathcal{E}$,

$$\mathcal{H}_\Gamma(x, v, \vec{p}) = \mathcal{H}(x, v, \vec{p}) - p^\mu_A \Gamma^A_\mu(x, v). \tag{36}$$

Here we have used that every connection in $\mathcal{E}$ can be interpreted as a map $\mathcal{E} \to \mathfrak{J}^1\mathcal{E}$. Furthermore, with the help of the volume form ω on $\mathcal{M}$ for every connection Γ there is a special function $\mathfrak{p}_\Gamma$ on $\mathcal{P}$ which uses that points in $\mathcal{P}$ are mappings of the image of the connection Γ. In coordinates,

$$\mathfrak{p}_\Gamma(x, v, \vec{p}, p) = p^\mu_A \Gamma^A_\mu(x, v) + p. \tag{37}$$

Combining these two, one obtains a function H that is independent of Γ,

$$H(x, v, \vec{p}, p) = -\mathcal{H}_\Gamma(x, v, \vec{p}) - \mathfrak{p}_\Gamma(x, v, \vec{p}, p) = -\mathcal{H}(x, v, \vec{p}) - p. \tag{38}$$

3.3. Hamiltonian n-vector fields on $\tilde{\mathcal{P}}$

One might ask whether a Hamiltonian n-vector field on $\tilde{\mathcal{P}}$ can be decomposable as well. We will show that this is not the case for typical examples. For simplicity we shall assume that the fibre bundle $\mathcal{E}$ admits a vanishing connection.

Again, we write a general ansatz for the n-vector fields that shall be combined to give a Hamiltonian n-vector field.

$$\tilde{Z}_\mu = \partial_\mu + (\tilde{Z}_\mu)^A \partial_A + (\tilde{Z}_\mu)^\nu_A \partial^A_\nu. \tag{39}$$

An evaluation of the defining relation

$$(\tilde{Z}_1 \wedge \cdots \wedge \tilde{Z}_n) \lrcorner \Omega_\Gamma = \mathrm{d}_\Gamma \tilde{H} \tag{40}$$

for some function $\tilde{H}$ yields no condition on the ∂_μ-components and the usual ones on the terms containing one vertical vector, namely

$$\partial^A_\rho \tilde{H} = (\tilde{Z}_\rho)^A - \partial_A \tilde{H} = (\tilde{Z}_\rho)^\rho_A. \tag{41}$$

Comparing this to the DW equation (5) we conclude that $\tilde{H}$ is to be interpreted as the DW Hamiltonian.

When looking at the 2-vertical components one encounters a difference, because the $\mathrm{d}p \wedge \mathrm{d}^n x$-term is missing in Ω_Γ. Therefore, instead of (28) one has

$$0 = (\tilde{Z}_\mu)^A (\tilde{Z}_\nu)^\nu_A - (\tilde{Z}_\nu)^A (\tilde{Z}_\mu)^\nu_A = -\partial^A_\mu \tilde{H} \partial_A \tilde{H} - \partial^A_\mu \tilde{H} \, (\tilde{Z}_\mu)^\nu_A. \tag{42}$$

Now let $\tilde{H}$ be given by

$$\tilde{H}(x, v, \vec{p}) = \tfrac{1}{2} g_{\mu\nu} \eta^{AB} p_A^\mu p_B^\nu + V(x, v), \tag{43}$$

where the function V is arbitrary and g and η denote metrics on space–time and the fibre, respectively. We now have

$$0 = -g_{\mu\rho} \eta^{AB} p_B^\rho \partial_A V - g_{\nu\rho} \eta^{AB} p_B^\rho (\tilde{Z}_\mu)_A^\nu, \tag{44}$$

from which by the independence of the polymomenta p_A^μ and the invertibility of g and η it follows that

$$(\tilde{Z}_\mu)_A^\nu = -\delta_\nu^\mu \partial_A V. \tag{45}$$

But this is in contradiction to $(\tilde{Z}_\mu)_A^\mu = -\partial_A V$ unless $n = 1$ or $\partial_A V = 0$.

3.4. Integrability

In the preceding subsections we have seen that Hamiltonian 0-forms on $\mathcal{P}$ of the particular form

$$H(x, v, \vec{p}, p) = -\mathcal{H}(x, v, \vec{p}) - p, \tag{46}$$

where $\mathcal{H}$ plays the role of the DW Hamiltonian, admit decomposable n-vector fields which can be interpreted as distributions on $\mathcal{P}$. The remaining question is whether there is an integrable distribution among them. Of course, given a set of n-vector fields that span the distribution under consideration, by the theorem of Frobenius [10] one just needs to verify that the vector fields close under the Lie bracket. However, as we have learned from (26), one cannot assign to a given Hamiltonian 0-form H a decomposition $X_H = Z_1 \wedge \cdots \wedge Z_n$ in a unique way. Rather, there is considerable arbitrariness in the choice of the polymomentum components $(Z_\mu)_A^\nu$. This has to be fixed in a satisfactory way. In this section, we will show that the required additional input comes from solutions of the covariant Hamilton–Jacobi equations.

Let us first examine the case of classical mechanics to understand the results below. In that case, to every time-dependent Hamiltonian there is a unique (time-dependent) vector field on the doubly extended phase space. Of course, this vector field can be integrated to yield a family of integral curves. However, the vector field cannot in general be projected onto the extended (covariant) configuration space $\mathbb{R} \times \mathcal{Q}$. Rather there is a correspondence between solutions of the Hamilton–Jacobi equation and set of curves on $\mathbb{R} \times \mathcal{Q}$. More precisely, one is looking for a map T that goes from $\mathbb{R} \times \mathcal{Q}$ to $\mathbb{R}^2 \times T^*\mathcal{Q}$ which pulls back the Hamiltonian vector field onto the extended configuration space. In the case of classical mechanics, this map happens to be the gradient of another function S. For the curves thus obtained to be solutions to the equations of motion, the function S needs to satisfy an additional equation, the celebrated Hamilton–Jacobi equation. In the simple case of classical mechanics this procedure is somewhat superfluous as it adds to the easy to handle set of ordinary differential equations a partial differential equation, but in the general case $n > 1$ this method turns out to be quite helpful.

Let us come back to the case of a higher dimensional base manifold $\mathcal{M}$. Here the fibre bundle $\mathcal{E}$ plays the rôle of the extended configuration space, while the extended multisymplectic phase space $\mathcal{P}$ replaces $\mathbb{R}^2 \times T^*\mathcal{Q}$. The desired map $T : \mathcal{E} \to \mathcal{P}$, cf. the diagram (12), has to possess two properties. Firstly, there should be an integrable distribution on $\mathcal{E}$ which is the pull back of some Hamiltonian n-vector field to the given function H. Secondly, the integral manifolds have to be solutions to the DW equations. Our aim will be to give necessary and sufficient conditions on T for the resulting set of integral submanifolds to be (local) foliations of $\mathcal{E}$. This constitutes, of course, the best possible case, and for general DW Hamiltonians one will have to lower one's sights considerably. In this paper, however, we are aiming at some geometrical picture and will, therefore, leave those matters aside.

Theorem 3. *Let $\mathcal{H}$ be a regular DW Hamiltonian. Then one can find a local foliation of $\mathcal{E}$ where the leaves (when transported to $\mathcal{P}$ by virtue of the covariant Legendre map (11)) are solutions of the DW equations if and only if there is a map $T : \mathcal{E} \to \mathcal{P}$ that satisfies in some coordinate system*

$$\partial_\mu^A \mathcal{H}(x, v, \vec{T}(x, v)) = 0, \tag{47}$$

$$\partial_\mu T_A^\mu(x, v) = -\partial_A \mathcal{H}(x, v, \vec{T}(x, v)), \tag{48}$$

$$\partial_\mu T_0(x, v) = -\partial_\mu \mathcal{H}(x, v, \vec{T}(x, v)), \tag{49}$$

$$\partial_\mu T_A^\mu(x, v) = -\partial_A T_0(x, v) \tag{50}$$

for all points (x, v) in a local neighbourhood of $\mathcal{E}$. Here, $\vec{T} = (T_A^\mu)$ denotes the p_A^μ-components of the map T while T_0 stands for the value of the p-component of T.

Remark. If the map T can be written as a derivative with respect to the field variables v^A of a collection of functions S^μ, $\mu = 1, \ldots, n$,

$$T_A^\mu(x, v) = (\partial_A S^\mu)(x, v), \qquad T_0(x, v) = (\partial_\mu S^\mu)(x, v) \tag{51}$$

then the second set of Eqs. (48) and (49), is a consequence of the generalised Hamilton–Jacobi equation for the functions S^μ (cf. [15, Chapter 4, Section 2]),

$$\partial_\mu S^\mu(x, v) + \mathcal{H}(x, v, \partial_A S^\mu(x, v)) = 0. \tag{52}$$

Clearly for $n = 1$ the sum in the first term reduces to the ("time") derivative of some function S, and this equation becomes the Hamilton–Jacobi equation. Note that the right-hand side of the second equation of (48) does not transform properly under a change of coordinates. This corresponds to the fact that if one chooses a different trivialisation, then the solutions to the DW equations will not be constant anymore. In other words, the transformed map T will not satisfy the generalised Hamilton–Jacobi equations.

Proof of the theorem. Let $\mathcal{U}$ be an open subset of $\mathcal{M}$ such that there is a local foliation of $\mathcal{E}$, i.e. a bijective map

$$\varphi : \mathcal{V} \times \mathcal{U} \to \mathcal{E} \restriction \mathcal{U}, \tag{53}$$

where $\mathcal{V}$ denotes the typical fibre of $\mathcal{E}$. This defines a local trivialisation of $\mathcal{E}$ which will be used for coordinate expressions for the rest of the proof. Furthermore, one obtains a map $T : \mathcal{E} \to \mathfrak{J}^1\mathcal{E} \to \mathcal{P}$ by taking the first jet prolongation of the section $\varphi(v, \cdot)$ for every point $\varphi(v, x)$ and transporting (via the Legendre map) this to $\mathcal{P}$. From

$$(\partial^A_\mu \mathcal{H})(x^\mu, v^A, (\partial^\mu_A L)(x^\mu, v^A, v^A_\mu)) = v^A_\mu,$$

where v^A_μ gives the value of the derivative w.r.t. the μ-direction when evaluated on sections, one concludes the first property. The remaining set of equations then follows from the fact the $\varphi(v, \cdot)$ are solutions to the DW equations for every element $v \in \mathcal{V}$ of the typical fibre.

Conversely, let T be a map which fulfils the conditions of the theorem. Then one can pull back a given decomposition of every Hamiltonian vector field of H to $\mathcal{E}$. Note that the resulting vector fields $\tilde{Z}_\mu$ are unique once the horizontal component of the Hamiltonian n-vector field has been fixed as in (23). From (26) one concludes that the resulting vector fields are horizontal in the chosen coordinate system. Therefore, they are integrable. Let

$$Z_\mu(x, v, T^\nu_A(x, v), T_0(x, v)) = \partial_\mu + \partial_\mu T^\nu_A(x, v)\partial^A_\nu - \partial_\mu \mathcal{H}(x, v, \vec{p})\partial, \tag{54}$$

$\mu = 1, \ldots, n$, be n-vector fields on the image of $\mathcal{E}$ under T (T_0 denotes the p-component of the map T). Then, comparing the second set of conditions to the second set of equations in (25), it follows by virtue of (48) and (49) that $Z_1 \wedge \cdots \wedge Z_n$ is indeed a Hamiltonian n-vector field to $H_{(x,v,\vec{p},p)} = -\mathcal{H}_{(x,v,\vec{p})} - p$. Furthermore, as the tangent vectors $\tilde{Z}_\mu$ on $\mathcal{E}$ do not have vertical components in this coordinate system, their integral surfaces cannot intersect. Hence, they describe a local foliation of $\mathcal{E}$.

Finally, having transported the sections from $\mathcal{E}$ via T to $\mathcal{P}$, their p-components by (49) and (50) can differ from $-\mathcal{H}$ only by a constant. □

Remark. The extended multisymplectic phase space can be identified with those n-forms on $\mathcal{E}$ that vanish upon contraction with two vertical (w.r.t. the projection onto $\mathcal{M}$) tangent vectors on $\mathcal{E}$. In coordinates, one has

$$(x^\mu, v^A, p^\mu_A, p) \cong p^\mu_A \, \mathrm{d}v^A \wedge \mathrm{d}_\mu x + p \, \mathrm{d}^n x. \tag{55}$$

Hence, the map T can be interpreted as an n-form on $\mathcal{E}$, and Eq. (51) can be interpreted as

$$T = \mathrm{d}S, \tag{56}$$

while (52) becomes

$$H \circ \mathrm{d}S = 0. \tag{57}$$

The conditions (48)–(50) now can be stated as

$$\mathrm{d}(H \circ T) = 0, \qquad \mathrm{d}T = 0. \tag{58}$$

3.5. An example: the free massive Klein–Gordon field

To conclude this paper, we will give an example to show that the assumptions of Theorem 3 are non-empty.

Let L be the Lagrange function of the Klein–Gordon field, i.e. let $\mathcal{E}$ be a trivial line bundle over $\mathcal{M} = \Sigma \times \mathbb{R} = \mathbb{R}^4$ and

$$L(x, v, v_\mu) = \tfrac{1}{2} g^{\mu\nu} v_\mu v_\nu - \tfrac{1}{2} m^2 v^2, \tag{59}$$

where $g^{\mu\nu}$ denotes the metric tensor. The Euler–Lagrange equation in this case is the celebrated Klein–Gordon equation

$$(\Box + m^2)\Phi(\vec{x}, t) = 0. \tag{60}$$

As is well known, for every pair of functions $\varphi, \pi \in \mathcal{C}^\infty(\Sigma)$ there is a unique function $\Phi \in \mathcal{C}^\infty(\Sigma \times \mathbb{R})$ given by convolution with certain distributions Δ, $\dot{\Delta}$,

$$\Phi(\vec{x}, t) = (\Delta * \pi_0)(\vec{x}, t) + (\dot{\Delta} * \varphi_0)(\vec{x}, t), \tag{61}$$

that satisfies the Klein–Gordon equation (60) and matches with the initial data φ, π:

$$\Phi(\vec{x}, 0) = \varphi(\vec{x}), \qquad (\partial_t \Phi)(\vec{x}, 0) = \pi(\vec{x}). \tag{62}$$

The corresponding DW Hamiltonian to L is given by

$$\mathcal{H}(x, v, p^\mu) = \tfrac{1}{2} g_{\mu\nu} p^\mu p^\nu + \tfrac{1}{2} m^2 v^2. \tag{63}$$

Let φ, π be a pair of initial data and Φ be the corresponding solution. The set of functions S^μ on $\mathcal{E}$ defined by

$$S^\mu(x^\mu, v) = v g^{\mu\nu}(\partial_\mu \Phi)(x) - \tfrac{1}{2}\Phi(x) g^{\mu\nu}(\partial_\nu \Phi)(x). \tag{64}$$

Clearly the S^μ satisfy

$$\begin{aligned} &(\partial_{p^\mu}\mathcal{H})(x, \Phi(x), \partial_\nu S^\mu(x, \Phi(x))) = g^{\mu\nu}(\partial_\nu \Phi)(x), \\ &(\partial_\mu S^\mu)(x, \Phi(x)) = -\mathcal{H}(x, \Phi(x), \partial_\nu S^\mu(x, \Phi(x))). \end{aligned} \tag{65}$$

Therefore,

$$\begin{aligned} X_\mu(x, v) = \partial_\mu + \partial_\mu \Phi(x)\partial_v + ((\partial_\mu S^\nu)(x, \Phi(x)) + (\partial_\mu \Phi)(x)(\partial_v S^\nu)(x, \Phi(x)))\partial_{p^\nu} \\ - (\partial_\mu \mathcal{H} + \partial_v \mathcal{H}\, \partial_\mu \Phi(x) + \partial_{p^\nu}\mathcal{H}\, \partial_\mu S^\nu + \partial_{p^\nu}\mathcal{H}\, \partial_v S^\nu\, \partial_\mu \Phi(x))\partial_p \end{aligned} \tag{66}$$

is a decomposition of a Hamiltonian 4-vector field of $H(x, v, p^\mu, p) = -\mathcal{H}(x, v, p^\mu) - p$.

4. Conclusions

We have clarified how n-dimensional submanifolds can be described by decomposable n-fold antisymmetrised tensor products of vector fields. Those multivector fields arise naturally in the context of multisymplectic geometry, cf. Eq. (14). The corresponding Hamiltonian forms are functions on the extended multisymplectic phase space $\mathcal{P}$. If such a Hamiltonian function is of the special form

$$H(x, v, \vec{p}, p) = -\mathcal{H}(x, v, \vec{p}) - p, \tag{67}$$

then is admits a decomposable Hamiltonian n-vector by Theorem 1.

Conversely, if one is given a solution to the DW equations with Hamiltonian $\mathcal{H}$, then its associated multivector field is Hamiltonian for the function (67). The p-dependence characterises the orientation of the solution submanifold as compared to the orientation on the base manifold $\mathcal{M}$. Its origin can be understood in a geometrical way.

Thirdly, given a DW Hamiltonian function (67), under certain additional conditions which use a generalisation of the Hamilton–Jacobi theory of classical mechanics, one can find an integrable Hamiltonian vector field on some subset of the extended multisymplectic phase space. This multivector field foliates the original fibre bundle where the theory has been formulated on. However—in contrast to the case of mechanics—one does not have a unique local foliation of the extended multisymplectic phase space $\mathcal{P}$ by solutions of the DW equations: even for the mass free scalar wave equation one can have two different solutions that coincide at one point with all their first derivatives, i.e. polymomenta.

The question of integrability does not arise in classical mechanics as there the equations of motion are ordinary differential equations.

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Appendix A. Distributions and multivectors

Much of this section seems to folklore by now. We add this material for the sake of completeness. It can be found for instance in [3,12]. Usually [10], when considering foliations of a given manifold $\mathcal{M}$, one introduces the notion of distributions, i.e. the determination of a subvector space of $T\mathcal{M}$ at every point of $\mathcal{M}$. Those subvector space can be described by specifying a basis at every point. This is somewhat ambiguous, but the antisymmetrised tensor product of the chosen basis is unique up to a pre-factor (the determinant of the basis transformation). On the other hand, in multisymplectic geometry, the concept of Hamiltonian k-vectors naturally arises, so it is plausible to examine the correspondence between distributions and multivectors.

Lemma A.1. *Let V be an $(n+m)$-dimensional vector space over some field $\mathbb{K}$ and X an element of the nth antisymmetric tensor product of V, $X \in \Lambda^n V$. Then there are n linearly independent vectors $\{Y_i\}_{i=1,\dots,n}$ that satisfy*

$$Y_i \wedge X = 0$$

if and only if

$$X = \lambda Y_1 \wedge \cdots \wedge Y_n,$$

where λ *is some element of* $\mathbb{K}$.

Proof. For the proof, one chooses a basis of V the contains the given Y_i. Then every n-vector X can be expanded in that basis, and one can successively show that all components containing the extra basis elements must vanish. □

Obviously, there cannot be more than n linearly independent vectors annihilating X. For if there were, one would have

$$0 \neq Y_1 \wedge \cdots Y_{n+1} = X \wedge Y_{n+1} = 0,$$

which is a contradiction.

There are, however, special cases apart from the trivial case $X \in \Lambda^{\max} V$, when the property of being decomposable is always fulfilled. Namely, let X be in $\Lambda^k V$ for $k = \dim V - 1$. Let $g(\cdot, \cdot)$ be a scalar product on V and $*$ be the corresponding Hodge star operation. Then,

$$\xi = *(X) \in V. \tag{A.1}$$

Let η_i be a basis of the orthogonal complement of ξ. Obviously

$$0 = g(\xi, \eta_i) = *^{-1}(\eta_i \wedge *\xi) = *^{-1}(\eta_i \wedge X), \tag{A.2}$$

hence $\eta_i \wedge X = 0$. From the lemma, we conclude that X is the antisymmetrised tensor product of all η_i. This case corresponds to the situation in three dimensions. There, planes can be described by 2 linearly independent vectors (which is ambiguous) or by indicating the vector perpendicular to the plane (which is unique up to a pre-factor). The latter can be understood as the Hodge dual (w.r.t. the scalar product that defines orthogonality) of the tensor product of the former two.

On the other hand, let $V = \mathrm{span}\{e_1, e_2, e_3, e_4\}$ and let $X = e_1 \wedge e_2 + e_3 \wedge e_4$. One can easily check that indeed there is no non-zero vector v that annihilates X, i.e.

$$X \wedge v = 0 \quad \Leftrightarrow v = 0. \tag{A.3}$$

Now we are in the position to formulate in terms of multivector fields the condition of a distribution E on $\mathcal{M}$ to be integrable. A distribution is integrable if every point of $\mathcal{M}$ belongs to some integral manifold of E. Let the distribution E be spanned by a set $\mathcal{W}$ of vector fields on $\mathcal{M}$ at every point. Then [10, Theorem 3.25] E is integrable if $\mathcal{W}$ is involutive, i.e. is closed under the Lie bracket of vector fields, and if E is of constant rank along the flow lines of all the vector fields of $\mathcal{W}$. Conversely, the tangent vectors of a given submanifold define local vector fields that span a distribution of constant rank and which are in involution.

Lemma A.2. *Let* X_E *be a multivector field that is associated with a* k*-dimensional distribution* E *on some manifold* $\mathcal{M}$*. Then* E *is integrable if and only if there are* k *linearly independent local vector fields* X_i *that satisfy*

$$[X_i, X] = \lambda_i X, \quad \lambda_i \in \mathcal{C}^\infty(\mathcal{M}), \tag{A.4}$$

where $[\cdot,\cdot]$ *denotes the Schouten bracket, which is a extension of the Lie bracket of vector fields* [17]. *For decomposable* n*-vectors, it is given by*

$$[X, Y] = \sum_{i=1}^{p}\sum_{j=1}^{q}(-)^{i+j}[X_i, Y_j] \wedge X_1 \wedge \cdots \widehat{X_i} \cdots X_p \wedge Y_1 \wedge \cdots \widehat{Y_j} \cdots Y_q. \tag{A.5}$$

Proof. Using (A.5) one verifies that $[X_i, X] = \lambda_i X$ iff $[X_i, X_j] = f_{ij}^k X_k$, but the latter condition means that the collection of all X_i define a distribution which is stable under the involutive closure of the X_i. □

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